Introduction to Mathematics and Modeling

lecture 8

Integration by parts

UNIVERSITY OF TWENTE.

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This week

intro

1. Section 8.1: integration by parts
2. Section 8.2: trigonometric integrals

Integrating the product rule

1.1

$$\boxed{\int f'(x)g(x)\ \mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\ \mathrm{d}x}$$

Proof:

- Recall the product rule for differentiation:
  $$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$
- Integrating both sides gives
  $$\begin{aligned}\int \frac{d}{dx}f(x)g(x)\ \mathrm{d}x &= \int f'(x)g(x) + f(x)g'(x)\ \mathrm{d}x\\ f(x)g(x) &= \int f'(x)g(x)\ \mathrm{d}x + \int f(x)g'(x)\ \mathrm{d}x.\end{aligned}$$
- Moving the second term to the left gives the boxed formula.

Integrating the product rule

1.2

For definite integrals the rule reads as

$$\begin{aligned}\int_a^b f'(x)g(x)\ \mathrm{d}x &= [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)\ \mathrm{d}x\\ &= f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\ \mathrm{d}x.\end{aligned}$$

Example 1, alternative 1

1.3

$$\int xe^x \, \mathrm{d}x = ??$$

Example 1: choose wisely...

1.4

$$\int xe^x \, \mathrm{d}x = ??$$

Explicit renaming

$$\boxed{\int f'(x)g(x)\ \mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\ \mathrm{d}x}$$

Notate $f(x) = u$ and $g(x) = v$, then the rule becomes

$$\boxed{\int u'v\ \ \mathrm{d}x = uv - \int u\,v'\ \ \mathrm{d}x}$$

Example 1, alternative 2

$$\int xe^x\ \mathrm{d}x = ??$$

Example 2

$$\int x^2 e^{-x} \, \mathrm{d}x = ??$$

Example 3

$$\int x \ln(x) \, \mathrm{d}x = ??$$

Example 4

$$\int \ln(x)\ \mathrm{d}x = ??$$

Implicit integration by parts

- With the explicit renaming $f(x) = u$ and $g(x) = v$:

$$\boxed{\int u'v\ \mathrm{d}x = uv - \int u\,v'\ \mathrm{d}x}$$

- Note that $du = u'\ \mathrm{d}x$ and $dv = v'\ \mathrm{d}x$. Therefore the rule can be memorized as follows:

$$\boxed{\int v\ \mathrm{d}u = uv - \int u\ \mathrm{d}v}$$

- You can even do this *without* renaming f and g:

$$\boxed{\int g(x)\ \mathrm{d}f(x) = f(x)g(x) - \int f(x)\ \mathrm{d}g(x)}$$

Abuse of the differential

recap 2.2

- If $u = g(x)$ then

$$du = g'(x)\,dx.$$

- Write

$$d(g(x)) = g'(x)\,dx.$$

- From right to left: *differentiate*, from left to right: *integrate*:

$d(x^2+1)$	$d(\frac{1}{3}x^3)$	$d(e^{2x})$
$2x\,dx$	$x^2\,dx$	$2e^{2x}\,dx$

- You may add an arbitrary constant to the right hand side:

$$2x\,dx = d\,x^2 = d\,(x^2+36).$$

Example 5

2.3

$$\int (2x+1)e^x \,\mathrm{d}x = ??$$

Example 6, first attempt

$$\int x^3 e^{x^2}\, \mathrm{d}x = ??$$

Example 6, alternative 1

$$\int x^3 e^{x^2}\, \mathrm{d}x = ??$$

Example 6, alternative 2

2.6

$$\int x^3 e^{x^2}\ \mathrm{d}x = ??$$

Example 7

2.7

$$\int e^{\sqrt{x}}\ \mathrm{d}x = ??$$

Example 8

$$I = \int e^x \cos(x)\, \mathrm{d}x = ??$$

Example 9

$$\int \cos(\ln x)\, \mathrm{d}x = ??$$

Powers of sines and cosines

4.1

Let m and n be non-negative integers.

$$\int \sin^m x \cos^n x \, \mathrm{d}x = ??$$

The following formulas are useful:

- $\sin^2 x + \cos^2 x = 1$
- $\sin x \cos x = \frac{1}{2} \sin(2x)$
- $\sin^2 x = \frac{1}{2}\big(1 - \cos(2x)\big)$
- $\cos^2 x = \frac{1}{2}\big(1 + \cos(2x)\big)$

Example 10, alternative 1

4.2

$$\int \sin^2 x \cos^2 x \, \mathrm{d}x = ??$$

Example 10, alternative 2

$$\int \sin^2 x \cos^2 x \; \mathrm{d}x = ??$$

Example 11

$$\int \cos^4 x \; \mathrm{d}x = ??$$

Example 12

4.5

$$\int \cos x \sin^2 x \ \mathrm{d}x = ??$$

Trigonometry: example 14

4.7

$$\int \cos x \ \sin(2x) \ \mathrm{d}x = ??$$